Revisiting Equivalence Relations

Recall: A relation $\sim$ on set S is an **equiv relation** when $\sim$ is reflexive, symm, and trans.

Ex: ① Equality is always an equiv relation

② $x \sim y$ when $x \equiv y \bmod n$ is an equiv. reln.

Defn: Let $\sim$ be an equiv relation on S. The **equivalence classes** of $\sim$ are

$$[x]_\sim := \{y \in S : x \sim y\}. \quad \text{for } x \in S.$$

↑ often omitted in notation...

Ex: If $\sim$ is equality on $\{a, b, c\}$, then

$$[a] = \{a\}, \quad [b] = \{b\}, \quad [c] = \{c\}.$$

Ex: (check) the following depicts an equiv relation:

Now $[a] = \{a, b\}$, $[c] = \{c\}$.

Ex: For equivalence mod 5 on $\mathbb{Z}$:

$$[0] = \{y \in \mathbb{Z} : 0 \equiv y \bmod 5\}$$
$$= \{y \in \mathbb{Z} : y = 5k + 0 \text{ for some } k \in \mathbb{Z}\}$$
$$= \{5k : k \in \mathbb{Z}\} = \{y \in \mathbb{Z} : 5 \mid y\}$$

In general: $[r] = \{5q + r : q \in \mathbb{Z}\}$. ↳ $[-1]$

Point: Five equivalence classes: $[0], [1], [2], [3], [4]$, $[5]$ ↰

Observations (Exercises for next homework):

If $\sim$ is an equivalence relation on S:

① $x \in [x]$ for all $x \in S$.

② $x \in [y]$ if and only if $y \in [x]$ for all $x, y \in S$.

③ If $[x] \cap [y] \neq \emptyset$, then $[x] = [y]$ for all $x, y \in S$.

The set of equivalence classes of $\sim$ forms a partition of S.

Exercise: Let $\mathcal{P}$ be a partition of S. There is a unique equivalence relation $\sim$ such that the equivalence classes of $\sim$ are precisely the elements of $\mathcal{P}$.

↳ Really good exercise to try...

NB: A covering set (or weak partition) can have empty elements (e.g. $\{\emptyset, \{1,2,3\}\}$ is a covering set for $[3]$.) Elements can also have nonempty intersection (e.g. $\{\{1,2\}, \{2,3\}\}$ is a covering set for $[3]$), whereas partitions can have neither of these...

Defn: A partition of set S is a collection $\mathcal{P} \subseteq \mathcal{P}(S)$ such that the following hold.

① $\emptyset \notin \mathcal{P}$

② For all $A, B \in \mathcal{P}$, if $A \cap B \neq \emptyset$, then $A = B$.

③ $\bigcup_{A \in \mathcal{P}} A = S$.

NB: Partitions are super-duper important in maths (because equivalence relations are) ☺

Functions

Q: When is a function an equivalence relation?

A: Exercise !!

Recall: Function is a relation $A \xrightarrow{f} B$ s.t. every element of A is related to a unique element of B.

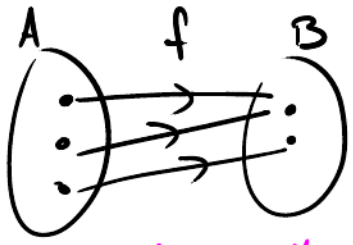
A function $f : A \to B$ is injective when

$$\forall_{x,y \in A} \left[f(x) = f(y) \longrightarrow x = y \right].$$

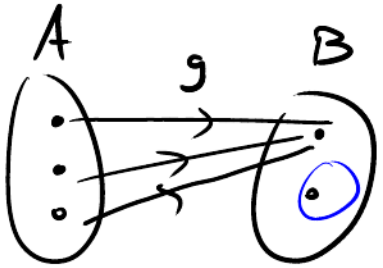
Defn: A function f is surjective when

$$\forall y \in \text{cod}(f) \; \exists x \in \text{dom}(f) \left[f(x) = y \right].$$

Ex:



Surjective !!



NOT Surjective !!

Ex: The function $f : \mathbb{Z} \to \mathbb{Z}$ via $f(n) = n - 7$ is surjective. Indeed, given $y \in \text{cod}(f) = \mathbb{Z}$, Note $y + 7 \in \text{dom}(f) = \mathbb{Z}$. Moreover $f(y+7) = (y+7) - 7 = y$. Hence for $x = y + 7$ we see $f(x) = y$. Hence f is surjective. ☑

Ex: The function $f : \mathbb{Z} \to \mathbb{N}$ via $f(n) = |n|$ is also surjective. For all $n \in \mathbb{N}$, we note $n \in \mathbb{Z}$, and $n \geq 0$, so $f(n) = |n| = n$, proving the claim. ☑

Ex: Is the function $f: \mathbb{Z} \to \mathbb{Z}$ via $f(n) = |n|$ surjective?

No! $|x| \geq 0$ for all $x \in \mathbb{Z}$, so $-1 \neq f(x)$ for all $x \in \mathbb{Z}$!

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Defn: The range of function f is the set

$$\text{ran}(f) := \{ y \in \text{cod}(f) : y = f(x) \text{ for some } x \in \text{dom}(f) \}$$

Prop: A function f is surjective if and only if $\text{ran}(f) = \text{cod}(f)$.

Defn: Given $S \subseteq \text{dom}(f)$ for function f, we define

$f(S) = \{ f(s) : s \in S \}$ is the image of S under f.

Preview of Coming Attractions:

Defn: Given $T \subseteq \text{cod}(f)$ for function f,

$f^{-1}(T) = \{ x \in \text{dom}(f) : f(x) \in T \}$ is the preimage of T under f.